Some Theoretical Issues in Sustainable Development: an Exposition*

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Abstract

We bring together a number of results on themes of continuing interest in the theory of sustainable development in the presence of natural resources. We rely on a discrete time approach. We present a partial equilibrium analysis of optimal resource extraction. Next we turn to multisector models and characterize the Malinvaud price of a resource. We examine the question of sustaining a positive consumption and conclude with an analysis of transition to a technology in which the natural resource does not constrain production.

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1 Introduction

In this paper we examine a number of themes of enduring interest in the studies of sustainable development in the presence of exhaustible resources. We begin with a complete (partial equilibrium) analysis of the patterns of optimal extraction of an exhaustible resource. We identify conditions under which there is a unique optimal program, provide a characterization of the support price (when the return function satisfies the Inada Condition, the optimal program is "interior," i.e., extraction is positive in all periods), and study the sensitivity of the initial optimal decision with respect to a change in the discount factor. When the return function satisfies a finite steepness property (often used, e.g., by Gale, 1967), the entire stock is exhausted in *finite* time. Next, we turn to a multisector model and identify a fundamental property of the competitive or Malinvaud price of the resource. We show that in the absence of appropriate input substitution (when the technology is a polyhedral cone) there is *no* program that can sustain a positive consumption. In the last section, we consider the problem of switching from a resource dependent technology to one in which the exhaustible resource does not impose any constraint on production.

There are many papers dealing with issues *similar* to the ones in this paper. Most of these use continuous time models, and rely on calculus of variations and/or the maximum principle as the primary technique of analysis. We use discrete time models, and our approach leads to elementary arguments using a few well-known results from real analysis and convex set theory. To our knowledge Proposition 11, which shows the impossibility of sustaining a positive level of consumption in the "stock version" of a multisector model of production, and the material in the last section are not available in the literature. We have made an attempt to provide an exposition that is largely self-contained and is of pedagogical interest.

No attempt is made to review the huge (related) literature (see Dasgupta and Heal, 1979, Conrad and Clark, 1978 and Brauer and Castillo-Chavez, 2001). The papers by Mitra (1978a), Mitra (1978b) and Cass and Mitra (1991) contain supplementary material on the efficient intertemporal allocation of a resource and input-substitutions needed to sustain indefinite positive consumption level in discrete time aggregative models.

2 Optimal Extraction of an Exhaustible Resource

2.1 Notation

We begin by introducing some notation. We say that a real sequence $\mathbf{x} = (x_t)$ is *non-negative,* written $\mathbf{x} \geq 0$, *when* $x_t \geq 0$ for all $t \geq 0$; we say $\mathbf{x}$ is *positive*, written $\mathbf{x} > \mathbf{0}$, *when* $\mathbf{x} \geq 0$, and $x_t > 0$ for some t; we say $\mathbf{x}$ is *strictly positive,* written $\mathbf{x} \gg 0$, *when* $x_t > 0$ for all $t \geq 0$. Similarly, for vectors.

2.2 A Model

A resource is *exhaustible* if its stock cannot be augmented given the known technology. Let $S > 0$ be the initial stock of such an exhaustible resource. An *extraction (or, consumption) program* (briefly, a program) from S is a *non-negative* sequence $\mathbf{c} = (c_t)_{t=0}^{\infty}$ satisfying:

$$\sum\nolimits_{t=0}^{\infty} c_t \leq S.$$

Let $\mathcal{C}$ be the set of all programs from a fixed $S > 0$. The set $\mathcal{C}$ is non-empty and convex.[1] We often write a program as $\mathbf{c} = (c_t)$.

A program $\mathbf{c} = (c_t)_{t=0}^{\infty}$ from $S > 0$ is *intertemporally efficient* if there does *not* exist another program $\mathbf{c}' = (c'_t)_{t=0}^{\infty}$ from (the same) S such that

$$c'_t \geq c_t \text{ for all } t, \text{ and } c'_t > c_t \text{ for some } t.$$

The following characterization of efficient programs (a special case of Majumdar (1974)) is useful.

[1] $\mathcal{C}$ is a subset of the linear space of all summable sequences.

Proposition 1 *A program* $\mathbf{c} = (c_t)_{t=0}^{\infty}$ *from* S *is intertemporally efficient if and only if*

$$\sum_{t=0}^{\infty} c_t = S. \tag{1}$$

We shall use the relation (1) repeatedly.

2.3 Maximizing Discounted Returns

Extraction (or, consumption) of the resource generates return (or, utility). Formally, the *return* function $u : R_+ \to R$ is assumed to satisfy:

U.1. u *is continuous;* $u(0) = 0$.

U.2. u *is increasing:* $c > c\prime$ implies $u(c) > u(c\prime)$.

U.3. u is *strictly concave*: $u\left(\theta c + (1-\theta)c\prime\right) > \theta u(c) + (1-\theta)u(c\prime)$ where $0 < \theta < 1$ and $c \neq c\prime$.

We shall introduce additional assumptions as we move on. Alternative programs are evaluated according to the "sum" of returns.

First, we recall Gale's (1967) famous example in the "undiscounted" case illustrating the non-existance of an optimal program.

Example 1: (Gale, 1967) Fix an initial stock $S > 0$. *A program* $\mathbf{c} = (c_t)_{t=0}^{\infty}$ *from* S is said to *overtake* another program $\mathbf{c}\prime = (c_t')_{t=0}^{\infty}$ from S if there exists some T such that

$$\sum_{t=0}^{\tau} \left[u(c_t) - u(c_t')\right] \geq 0 \text{ for all } \tau \geq T.$$

A program $\mathbf{c} = (c_t)_{t=0}^{\infty}$ *from* S is *optimal (according to the "overtaking" criterion)* if it overtakes all programs from S. Clearly,the program of zero consumption (i.e., $c_t = 0$ for all t) cannot be optimal. In any other program $\mathbf{c} = (c_t)$, there must be two values c_T and c_{T+1} that are not equal. But in this case, we can construct a program $c\prime = (c_t')$ such that

$$\begin{aligned} c_t' &= c_t \text{ for } t \neq T,\ T+1 \\ c_T' &= c_{T+1}' = \frac{c_T + c_{T+1}}{2} \end{aligned}$$

By strict concavity of u we have:

$$u(c_T) + u(c_{T+1}) < 2u\left(\frac{c_T + c_{T+1}}{2}\right) = u(c'_T) \ + u(c'_{T+1})$$

So, $\mathbf{c}\prime = (c'_t)$ overtakes $\mathbf{c} = (c_t)$. Hence, there is no optimal program according to the "overtaking" criterion.

■

In view of this example, we focus on the "discounted" case: a discount factor δ is given, $0 < \delta < 1$. The "discounted" optimization model we study in detail is specified by $\{\mathcal{C}, S, u, \delta\}$. Observe that for *any* program $\mathbf{c} = (c_t)_{t=0}^{\infty}$ *from* $S > 0$*, the discounted sum of utilities is bounded* since we have

$$\sum_{t=0}^{\infty} \delta^t u(c_t) \leq \sum_{t=0}^{\infty} \delta^t u(S) \equiv \frac{u(S)}{1-\delta}.$$

In particular, given U.1., we shall use

$$\sum_{t=0}^{T} \delta^t u(c_t) \to \sum_{t=0}^{\infty} \delta^t u(c_t) \text{ as } T \to \infty. \tag{2}$$

A program $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ from $S > 0$ is *optimal* if

$$\sum_{t=0}^{\infty} \delta^t u(c_t^*) \geq \sum_{t=0}^{\infty} \delta^t u(c_t)$$

for *all* programs $\mathbf{c} = (c_t)_{t=0}^{\infty}$ from (the same initial stock) $S > 0$.

2.4 Optimal Programs

We begin by proving the existence and uniqueness of an optimal program.

Proposition 2 *Under U.1.-U.3., there is a unique optimal program.*

Proof. We provide a sketch, since the details can be adapted from Bhattacharya and Majumdar (2007, Theorem 9.1). Let $\mathbf{c} = (c_t)_{t=0}^{\infty}$ be a program from $S > 0$. Write

$$w(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t u(c_t)$$

and let

$$\alpha = \sup\{w(\mathbf{c}) : \mathbf{c} \in \mathcal{C}\}$$

Choose a sequence $\mathbf{c}^n = (c_t^n)$ in $\mathcal{C}$ such that

$$w(\mathbf{c}^n) \geq \alpha - \frac{1}{n}.$$

By a diagonalization argument there is a subsequence of $\mathbf{c}^n$ (retain notation) such that for all $t \geq 0$,

$$c_t^n \to c_t^* \text{ as } n \to \infty,$$

where $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ is some non-negative sequence. The proof is completed by verifying that (i) $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ is a program from S; and (ii) $w(\mathbf{c}^*) = \alpha$, so that it is indeed an optimal program. Uniqueness is a consequence of strict concavity, U.3. ■

It is clear that the optimal program is intertemporally efficient, i.e., $\sum_{t=0}^{\infty} c_t^* = S$.

For simplicity of exposition, we make the following assumption on u :

U.4. *u is twice continuously differentiable on R_{++}; $u'(c) > 0$ and $u''(c) < 0$ at $c > 0$.*

We now study the qualitative properties of the optimal program. We consider two cases, first with an unbounded marginal utility and then with a bounded marginal utility, separately.

Case 1. Here we make the assumption:

U.5. (Inada) $u'(c) \to \infty$ *as* $c \to 0$.

Under this assumption the unique optimal program $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ satisfies $c_t^* > 0$.

Proposition 3 *Under U.1.-U.5., if $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ is the optimal program from S, then there is a constant $\lambda^* > 0$ such that*

$$\delta^t u'(c_t^*) = \lambda^* \text{ for all } t \geq 0. \tag{3}$$

Proof. If $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ is the optimal program from S, and (by U.5.) $c_t^* > 0$, the condition (3) can be derived (given our differentiability assumption U.4.) from Peleg (1971, Theorem 3.9, Theorem 4.1 and Example 4.3). (See also the Support Theorems for l_1 in Cass and Majumdar (1979, pp. 230-231, and Section 6, pp. 244-246). The precise proof of Proposition 3 is too long for reproduction. ■

It is perhaps useful to think of (3) heuristically as follows: suppose that a strictly positive $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ is the optimal program from S. Then there is some multiplier $\lambda^* > 0$ such that $(\mathbf{c}^*, \lambda^*)$ also maximize the Lagrangian expression

$$\mathcal{L}\{(c_t), \lambda\} \equiv \sum_{t=0}^{\infty} \delta^t u(c_t) - \lambda[\sum_{t=0}^{\infty} c_t - S] \tag{4}$$

over the set of all non-negative sequences in the space of summable sequences. Setting the partials of the Lagrangian with respect to c_t all equal to zero at $\mathbf{c}^*$, one gets (3).

We should note a converse of Proposition 3. For the sake of completeness, we sketch a proof.

Proposition 4 *Under U.1.-U.5., let a strictly positive sequence* $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ *from* S *be an intertemporally efficient program such that for some* $\lambda^* > 0$ *the condition (3) holds. Then* $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ *is the optimal program from* S.

Proof. Take any program $\mathbf{c} = (c_t)_{t=0}^{\infty}$ from $S > 0$. Note that for any finite T,

$$\begin{aligned} & \sum_{t=0}^{T} \delta^t u(c_t) - \sum_{t=0}^{T} \delta^t u(c_t^*) \\ \leq \; & \sum_{t=0}^{T} [\delta^t (c_t - c_t^*) u'(c_t^*)] \text{ [use } u''(c) < 0 \text{ at } c > 0] \\ = \; & \sum_{t=0}^{T} (c_t - c_t^*) \lambda^* \leq \lambda^* (\sum_{t=0}^{T} c_t) - \lambda^* (\sum_{t=0}^{T} c_t^*). \end{aligned}$$

Looking at the last expression one notes that $\sum_{t=0}^{\infty} c_t \leq S$ and $\sum_{t=0}^{\infty} c_t^* = S$. Now taking the limit as $T \to \infty$, one gets (use (2)):

$$\sum_{t=0}^{\infty} \delta^t u(c_t) - \sum_{t=0}^{\infty} \delta^t u(c_t^*) \leq 0.$$

which completes the proof of optimality. ■

One immediately concludes from (3) that

$$\delta u'(c^*_{t+1}) = u'(c^*_t) \text{ for all } t \geq 0. \tag{5}$$

It follows that $c^*_t > c^*_{t+1}$, that is, in the optimal program consumption declines over time.

We shall state two comparative static results. First, we show that when the stock of the resource is higher, the optimal program involves higher consumption in every period. Second we show that if the consumer is more patient, he consumes less in the initial period.

Proposition 5 *Under U.1.-U.5., (i) Let $S > S' > 0$ and $\mathbf{c}^*(S) = (c^*_t(S))$ and $\mathbf{c}^*(S') = (c^*_t(S'))$ be the optimal programs from S and S' respectively. Then,*

$$c^*_t(S) > c^*_t(S') \text{ for all } t \geq 0.$$

(ii) Let $0 < \delta < \delta' < 1$. Fix $S > 0$. Let $\mathbf{c}^*(\delta) = (c^*_t(\delta))$ and $\mathbf{c}^*(\delta') = (c^*_t(\delta'))$ be the optimal programs from S, corresponding to δ and δ'respectively. Then,

$$c^*_0(\delta) > c^*_0(\delta'). \tag{6}$$

Proof. (i). This can be readily adapted from the proof of Lemma 9.2 of Bhattacharya and Majumdar (2007)[2]. (ii) Assume, to the contrary that

$$c^*_0(\delta) \leq c^*_0(\delta'). \tag{7}$$

This leads to

$$u'(c^*_0(\delta)) \geq u'(c^*_0(\delta')). \tag{8}$$

By using (5) and (8):

$$\delta u'(c^*_1(\delta)) = u'(c^*_0(\delta)) \geq u'(c^*_0(\delta')) = \delta' u'(c^*_1(\delta')). \tag{9}$$

[2] Although the model treated there is different.

Hence, the assumption that $\delta < \delta'$ and (9) lead to

$$u'(c_1^*(\delta)) > u'(c_1^*(\delta')),$$

which means

$$c_1^*(\delta) < c_1^*(\delta').$$

Repeating the step we get

$$c_t^*(\delta) < c_t^*(\delta') \text{ for all } t \geq 1.$$

Hence,

$$S = \sum_{t=0}^{\infty} c_t^*(\delta) < \sum_{t=0}^{\infty} c_t^*(\delta') = S$$

This contradiction means that (7) must be false, establishing the claim (6). ■

Case 2. Here we make the assumption:

U.6. (Gale) $\lim_{c \to 0} u'(c) \leq M$ *for some positive (finite)* M.

Proposition 6 *Consider the optimal program* $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$. *Under U.1.-U.4. and U.6., if for some finite* T, $c_T^* = 0$, *then* $c_{T+t}^* = 0$, *for all* $t \geq 1$.

Proof. Suppose that for some finite T, $c_T^* = 0$, and there is some $\tau > T$ such that $c_\tau^* > 0$. Then, by U.2. (monotonicity)

$$u(c_\tau^*) > u(c_T^*) = u(0).$$

Consider the program $\mathbf{c} = (c_t)$ defined as $c_t = c_t^*$ for all $t \neq T, \tau$; $c_T = c_\tau^* > 0$ and $c_\tau = c_T^* = 0$. Then, $\mathbf{c}$ is also a program from S because $\sum_{t=0}^{\infty} c_t = S$. The difference between the sum of utilities of the two programs is:

$$\begin{aligned}
\sum_{t=0}^{\infty} \delta^t u(c_t) - \sum_{t=0}^{\infty} \delta^t u(c_t^*) &= [\delta^T u(c_T) + \delta^\tau u(c_\tau)] - [\delta^T u(c_T^*) + \delta^\tau u(c_\tau^*)] \\
&= [\delta^T u(c_\tau^*) + \delta^\tau u(c_T^*)] - [\delta^T u(c_T^*) + \delta^\tau u(c_\tau^*)] \\
&= (\delta^T - \delta^\tau)[u(c_\tau^*) - u(0)] > 0,
\end{aligned}$$

which contradicts the optimality of $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$. ■

Proposition 7 *Let $\mathbf{c}^* = (c_t^*)_{t=0}^{\infty}$ be the optimal program. Under U.1.-U.4. and U.6., there is a finite T such that $c_t^* = 0$ for all $t > T$.*

Proof. If the claim is not true, $c_t^* > 0$ for all $t \geq 0$. We first argue that (3) holds also under the assumption of this proposition, that is for some $\lambda^* > 0$

$$\delta^t u'(c_t^*) = \lambda^* \text{ for all } t \geq 0.$$

Fix any finite $T \geq 1$.Write $s_T^* = s - \sum_{t=0}^{t=T} c_t^*$.Then $s_T^* > 0$ for all T. Now consider all non-negative $c = (c_0, c_1, ...c_T)$ such that

$$\sum_{t=0}^{t=T} c_t \leq s - s_T^*$$

By the optimaility principle, $\sum_{t=0}^{t=T} \delta^t u(\ c_t^*) \geq \sum_{t=0}^{t=T} \delta^t u(\ c_t)$. By the standard Lagrangian method, there is some λ_T^* such that

$$\delta^t u'(c_t^*) = \lambda_T^*$$

But choosing the horizon $T, T+1, ...$ we get

$$\delta^t u'(c_t^*) = \lambda_T^* = \lambda_{T+1}^* = ... = \lambda^*, \text{ say.}$$

But $u'(c_t^*) \leq M$ and $\delta^t \to 0$, so we get a contradiction. ■

Proposition 8 *Under U.1.-U.4. and U.6., the optimal program is $c_0^* = S$ and $c_t^* = 0$ for all $t > 0$ if and only if $u'(S) \geq \delta u'(0)$.*

Proof. If $c_0^* = S$ is an optimal program, then it must be that $u'(S) \geq \delta u'(0)$, else if $u'(S) < \delta u'(0)$ transferring a small enough ε from period 0 to period 1 would increase the value of the program. Conversely, suppose that $u'(S) \geq \delta u'(0)$ but $c_0^* = S$ is not an optimal program, then in the optimal program consumption is positive in at least the first two periods. Thus, $S > c_0^* > c_1^* > 0$. Because u is concave this implies that

$$u'(S) < u'(c_0^*) < u'(c_1^*) < u'(0).$$

Additionally, the first order conditions imply that $u'(c_0^*) = \delta u'(c_1^*)$. Hence,

$$u'(S) < u'(c_0^*) = \delta u'(c_1^*) < \delta u'(0).$$

This contradicts the assumption that $u'(S) \geq \delta u'(0)$. ■

3 Malinvaud Price of a Resource

At this point it is useful to recall the "stock version" of the well-known multisector model of production (see Nikaido, 1968 for a detailed exposition). Suppose that there are n (finite) commodities. The production possibilities open to an economy are described by a non-empty set $\mathcal{T}$ in R_+^{2n}. Formally,

$$\mathcal{T} = \{(x, y) \in R_+^{2n} : y \text{ is producible from } x\}. \tag{10}$$

We interpret y as an output vector resulting from the use of the input vector x. There is a lag of one period in the production process from one input vector x_t in period t, the output vector y_{t+1} appears in period $t+1$. We shall refer to $(x, y) \in \mathcal{T}$ as an *input-output pair*, or, simply an *activity*. In any input-output pair (x, y), x_j describes the quantity of commodity j used as an input and y_j the quantity of output of the same commodity. A commodity k is an *exhaustible resource* if "for *all* $(x, y) \in \mathcal{T}$, $y_k \leq x_k$"; a commodity j is a *renewable resource* if "$(x, y) \in \mathcal{T}$ and $x_j = 0$" implies that "$y_j = 0$". A commodity h is a *primary factor of production* if "$y_h = 0$ for all $(x, y) \in \mathcal{T}$ ". To clarify the use of these terms in the context of this model, let us stress the characteristic properties of these three types of commodities: the quantity of an exhaustible resource cannot be increased by using *any* activity; if the stock of a renewable resource is zero, it cannot be obtained in positive quantity by using *any* activity; a primary factor of production like "labor" is used as an input, but never appears in the output vector of *any* activity. To simplify exposition, we assume that there is at most one primary factor of production and one exhaustible resource, and write $x = (\tilde{x}, l, s) \geqq 0$ - where $\tilde{x}$ is a vector of producible goods and renewable resources, l is the quantity of labor and s is the stock of exhaustible resource. Similarly, write $y = (\tilde{y}, 0, \tilde{s}) \geqq 0$. Assume that neither the resource nor the primary factor is directly consumed.

We begin with the following assumptions:

T.1. *$\mathcal{T}$ is a convex cone and* $(0, 0) \in \mathcal{T}$.

The input-output vectors are listed so as to label the resource as the last commodity. Hence,

T.2. "$((\tilde{x}, l, s), (\tilde{y}, 0, \tilde{s})) \in \mathcal{T}$" *implies* $s \geq \tilde{s}$.

T.3. *For any* $(\tilde{x}, l, s) \geqq 0$, $((\tilde{x}, l, s), (\tilde{x}, 0, s)) \in \mathcal{T}$.

Assumption T.3. means that the technology allows free storage of producible capital goods as well as the resource.

T.4. "$((\tilde{x}, l, s), (\tilde{y}, 0, \tilde{s})) \in \mathcal{T}$" implies "$((\tilde{x}', l', s'), (\tilde{y}', 0, \tilde{s}')) \in \mathcal{T}$" where $(\tilde{x}', l', s') \geqq (\tilde{x}, l, s)$ and $0 \leq \tilde{y}' \leq \tilde{y}$, $0 \leq \tilde{s}' \leq \tilde{s}$.

Assumption T.4. is the standard free disposal assumption.

T.5. "$((\tilde{x}, l, s), (\tilde{y}, 0, \tilde{s})) \in \mathcal{T}$ *with* $r = s - \tilde{s} > 0$" *implies that for any* $\check{s} > r$, "$((\tilde{x}, l, \check{s}), (\tilde{y}, 0, \check{s} - r)) \in \mathcal{T}$"

Assumption T.5. means that in the activity, the actual resource used denoted by r determines whether one can transform the inputs $(\tilde{x}, l)$ into the output $(\tilde{y}, 0)$.

For any activity $(x, y) \equiv ((\tilde{x}, l, s), (\tilde{y}, 0, \tilde{s})) \in \mathcal{T}$, we write

$$r = s - \tilde{s} \geq 0$$

to denote the quantity of the resource used up by the activity (x, y).

The initial stock of producible goods $X \gg 0$, of the resource $S > 0$, and the supply of primary factor in every period t, $(\mathbf{l}_t) \gg 0$ are all given. Write $\mathbf{l} = (\mathbf{l}_t)$. A program is a sequence $(\mathbf{x}, \mathbf{y}, \mathbf{s}; \mathbf{c}, \mathbf{l}) = (x_t, y_{t+1}, s_t; c_{t+1}, l_t)$ satisfying

$$\begin{aligned} \tilde{x}_0 &= X \text{ , } s_0 = S, \; l_t = \mathbf{l}_t \text{ for } t \geq 0; \\ (x_t, y_{t+1}) &\equiv ((\tilde{x}_t, \; l_t, s_t), (\tilde{y}_{t+1}, 0, s_{t+1})) \in \mathcal{T} \text{ for } t \geq 0; \\ c_{t+1} &= \tilde{y}_{t+1} - \tilde{x}_{t+1} \geq 0 \text{ for } t \geq 0. \end{aligned} \tag{11}$$

Letting $r_t = s_t - s_{t+1} \geq 0$ we get, along any program (11) a sequence s_t of resource stocks and extractions r_t defined as follows:

$$s_0 = S; \; s_t = S - \sum_{t=0}^{t-1} r_t \text{ for } t \geq 1.$$

Since $s_t \geq 0$, we get

$$\sum_{t=0}^{\infty} r_t \leq S.$$

A program is *interior* if $(\tilde{x}_t, r_t) \gg 0$. Thus, for an interior program, it is also true (use the last line of (11)) that $(\tilde{y}_{t+1}) \gg 0$. Also, note that if $(r_t) \gg 0$, the corresponding $(s_t) \gg 0$.

A program $(\mathbf{x}, \mathbf{y}, \mathbf{s}; \mathbf{c}, \mathbf{l}) = (x_t, y_{t+1}, s_t; c_{t+1}, l_t)$ is called *competitive if* there is a sequence of price vectors $(\mathbf{p}, \mathbf{q}, \mathbf{w}) = (p_t.q_t, w_t)$ (where $\mathbf{p} > 0, \mathbf{q} > 0, \mathbf{w} > 0$) such that, for all $t \geq 0$,

$$\begin{aligned} & p_{t+1} \cdot \tilde{y}_{t+1} + q_{t+1} \cdot s_{t+1} - p_t \cdot \tilde{x}_t - q_t \cdot s_t - w_t \cdot l_t \\ \geq \quad & p_{t+1} \cdot \tilde{y} + q_{t+1} \cdot \tilde{s} - p_t \cdot \tilde{x} - q_t \cdot s - w_t \cdot l \;\text{ for all } (x, y) \equiv ((\tilde{x}, l, s), (\tilde{y}, 0, \tilde{s})) \in \mathcal{T} \end{aligned} \tag{12}$$

In other words, a program is competitive if the intertempral profit maximization condition (12) is satisfied relative to some positive sequence of "support" or "Malinvaud prices" in every period. An important property of the Malinvaud price sequence supporting the allocation of an exhaustible resource $\mathbf{q} = (q_t)$ is now stated and proved. It is surely a basic result in resource economics.

Proposition 9 *If an interior program* $(\mathbf{x}, \mathbf{y}, \mathbf{s}; \mathbf{c}, \mathbf{l})$ *is competitive at Malinvaud prices* $(\mathbf{p}, \mathbf{q}, \mathbf{w})$, *then* $q_t = q_{t+1}$ *for all* $t \geq 0$.

Proof. Since $\mathcal{T}$ is a cone, the left side of the inequality in (12), i.e., the maximum profit, must be zero for all $t \geq 0$. Since $((0, 0, s), (0, 0, s)) \in \mathcal{T}$ for $s > 0$, using (12) we get $0 \geq (q_{t+1} - q_t)s$, implying $(q_{t+1} - q_t) \leq 0$.

Now, suppose that for some period t, $q_{t+1} < q_t$. Since the program $(\mathbf{x}, \mathbf{y}, \mathbf{s}; \mathbf{c}, \mathbf{l})$ is interior, $s_t > r_t > 0$. Choose some s'' such that $s_t > s'' > r_t$. Then, surely, $((\tilde{x}_t, l_t, s''), (\tilde{y}_{t+1}, 0, s'' - r_t)) \in \mathcal{T}$ and using this on the right side in (12), we get (after cancellation of common terms on both sides)

$$
\begin{aligned}
q_{t+1} \cdot s_{t+1} - q_t \cdot s_t &\geq q_{t+1} \cdot (s'' - r_t) - q_t \cdot s'' \\
\text{or, } q_{t+1} \cdot (s_t - r_t) - q_t \cdot s_t &\geq q_{t+1} \cdot (s'' - r_t) - q_t \cdot s'' \\
\text{or, } (q_{t+1} - q_t) \cdot s_t &\geq (q_{t+1} - q_t) \cdot s'' \\
\text{or, } (q_{t+1} - q_t) \cdot (s_t - s'') &\geq 0 \qquad (13)
\end{aligned}
$$

Since $(q_{t+1} - q_t) < 0$ by assumption and $(s_t - s'') > 0$, we have a contradiction, establishing the claim that

$$q_t = q_{t+1} \text{ for all } t \geq 0.$$

■

Mitra (1978a) contains a definitive treatment of a number of issues involving the efficient allocation of an exhaustible resource, and the link between efficient and competitive programs.

3.1 Sustainable Consumption and Input Substitution

In contrast with the literature on "optimal" growth in the Ramsey-Koopmans tradition (where the optimality criterion is an appropriate sum of utilities from consumption), a significant volume of literature on exhaustible or renewable resource management concentrates on the issue of maintaining (or, sustaining) a positive level of consumption (or, harvesting or, extraction) in *every* period. Informally, the objective is to "survive" or minimize the possibility of extinction rather than to "optimize". Clearly, the target level set by the planner (or, the firm, presumably under the guidance of a planner) must take note of the *productivity* of the technology. We begin with a simple example to clarify this issue (see Majumdar and Radner, 1992 for a more elaborate analysis).

Example 2: Consider an economy which starts with an initial stock y of a renewable resource. In each period t, the economy is required to *consume* or *harvest* a positive amount c of the beginning of the period stock y_t. The remaining stock in that period $x_t = y_t - c$ is 'invested' and the resulting output (principal plus return) is the beginning of the period stock y_{t+1}. The output y_{t+1} is related to the input x_t by a 'production' (regeneration) function g. Assume that

A.1. $g : \mathbb{R} \to \mathbb{R}$ *is continuous, and increasing on* $\mathbb{R}_+$, $g(x) = 0$ *for* $x \leq 0$;

A.2. *there is some* $x^* > 0$ *such that* $g(x) > x$ *for* $0 < x < x^*$ *and* $g(x) < x$ *for* $x > x^*$.

A.3. g *is concave.*

The evolution of the system (given the initial $y > 0$ and the planned harvesting $c > 0$) is described by:

$y_0 = y$,

$x_t = y_t - c$, $t \geq 0$;

$y_{t+1} = g(x_t)$ for $t \geq 0$.

Let T be the first period t, if any, such that $x_t < 0$, if there is no such t, then $T = \infty$. If T is finite we say that the agent (or the resource) *survives up to* (but not including) period T. We say that the agent survives (forever) if $T = \infty$ (i.e., if $x_t \geq 0$ for all t).

Define the net return function $h(x) = g(x) - x$. It follows that h satisfies

$$\begin{cases} h(x) \geq 0 \text{ if } 0 < x < x^*; \\ h(x) = 0 \text{ if } x = 0, \text{ or if } x = x^*; \\ h(x) < 0 \text{ if } x > x^*. \end{cases}$$

Since $g(x) = 0$ for all $x \leq 0$, all statements about g and h will be understood to be for non-negative arguments unless something explicit is said to the contrary. Actually, we are only interested in following the system up to the 'failure' or 'extinction' time T.

The *maximum sustainable harvest or consumption* is

$$H = \max_{[0,x^*]} h(x).$$

We write

$$x_{t+1} = x_t + h(x_t) - c.$$

If $c > H$, $x_{t+1} - x_t = h(x_t) - c < H - c < 0$. Hence, x_t will fall below (extinction) 0 after a finite number of periods. On the other hand, if $0 < c < H$, there will be two roots ξ' and ξ'' of the equation

$$c = h(x)$$

which have the properties

$$0 < \xi' < \xi'' < x^*.$$

and

$$\begin{cases} h(x) - c \geq 0 \text{ if } \xi' < x < \xi''; \\ h(x) - c = 0 \text{ if } x = \xi' \text{ or if } x = \xi''; \\ h(x) - c < 0 \text{ if } x < \xi' \text{ or if } x > \xi''. \end{cases}$$

We can show that

a) If $x_0 < \xi'$, then x_t reaches or falls below 0 in finite time.

b) If $x_0 = \xi'$, then $x_t = \xi'$ for all t.

c) If $x_0 > \xi'$, then x_t converges monotonically to ξ''.

Note that if $c = H$, there are two possibilities: either $\xi' = \xi''$ (i.e., $h(x)$ attains the maximum H at a unique period ξ') or, for all x in a non-degenerate interval $[\xi', \xi'']$, $h(x)$ attains its maximum.

The implications of the foregoing discussion for survival and extinction are summarized as follows.

Proposition 10 *Let $c > 0$ be the planned consumption for every period and H the maximum sustainable consumption.*

1) If $c > H$, there is no initial y from which survival (forever) is possible.

2) If $0 < c < H$, then there is $\xi' > 0$, with $h(\xi') = c$

such that survival is possible if and only if the initial stock $y \geq \xi' + c$

3) If $c = H$, then there is $\xi' > 0$ with $h(\xi') = H$, and ξ' tends to 0 as c tends to 0.

■

It is intuitive that, when a technology uses both producible capital goods and natural resources as *essential* inputs, the prospects of meeting an exogenously specified target of consumption depends on the possibilities of substitution between (producible) capital goods and (non-producible) resources, enhancement of productivity and emergence of technologies less dependent on natural resources. We make a few comments on each of these themes . To see the role of substitution between producible capital goods and a resource, we go back to the stock version of the multisector model and consider the case in which there is no primary factor of production.[3] Thus, an activity or input-output pair is described by $(x, y) \equiv ((\tilde{x}, s), (\tilde{y}, \tilde{s}))$ with the understanding that $s \geq \tilde{s}$. We retain the assumptions T.2.-T.5. on the technology $\mathcal{T}$ (deleting all the references to the quantity l of the primary factor in the statements. We replace T.1. by:

T.1'. $\mathcal{T}$ is a *polyhedral convex cone* (there is a finite collection of activities $(a^1, b^1), ..., (a^v, b^v)$ in $\mathcal{T}$ such that if $(x, y) \in \mathcal{T}$, $(x, y) \equiv \sum_{k=1}^{v} (\lambda_k \cdot a^k, \lambda_k \cdot b^k)$ where $\lambda_k \geq 0$).

See Nikaido (1968, pp.41-43) for a review of the properties of polyhedral convex cones. It is known that a *polyhedral cone is also closed* (see Moore, 2007, pp.187-188).

The importance of the resource in production is captured by the following assumption:

T.6. *Let* $(x, y) \equiv ((\tilde{x}, s), (\tilde{y}, \tilde{s})) \in \mathcal{T}$ *and* $\tilde{y} > \tilde{x}$. *Then* $s > \tilde{s}$.

[3] Thus, to be consistent with (10) we are assuming $(n - 1)$ producible goods and one resource where $n \geq 2$.

In other words, for the net production of any producible good, a positive quantity of the resource is essential.

With the assumptions T.1'., T.2. through T.6., and with a given vector of initial stocks $X \gg 0$ of the producible goods and the initial stock $S > 0$ of the resource, we define a program $(\mathbf{x}, \mathbf{y}, \mathbf{s}; \mathbf{c}) = (x_t, y_{t+1}, s_t; c_{t+1})$ as in (11) above, but deleting all references to the primary factor of production. For a non-negative vector $c = (c_i)$ define the norm $\|c\|$ as

$$\|c\| = \sum_i c_i. \tag{14}$$

The basic result on the impossibility of sustaining a positive level of consumption is now stated and proved.

Proposition 11 *Assume T.1'., T.2. through T.6.. Let* $(\mathbf{x}, \mathbf{y}, \mathbf{s}; \mathbf{c}) = (x_t, y_{t+1}, s_t; c_{t+1})$ *be any program (given* $X \gg 0$, $S > 0$*). Then,*

$$\lim_{t \to \infty} \|c_t\| = 0.$$

Proof of Proposition 11. To prove the proposition we first need a preliminary lemma. Define $Z = \{z \in R^n : z = y - x, \ (x, y) \in \mathcal{T}\}$. Note that Z is a (polyhedral) closed convex cone. Indeed, any $z \in Z$ can be expressed as

$$z = \sum_{k=1}^{v} \lambda_k \cdot (b^k - a^k)$$

Lemma 1 Under assumptions T.1'., T.2. through T.6., there is $p \gg 0$ such that $p \cdot z \leq 0$ for all $z \in Z$.

Proof of Lemma 1. Consider any $z \in Z$. Then $z = y - x$, for some $(x, y) \in \mathcal{T}$. Write $(x, y) \equiv ((\tilde{x}, s), (\tilde{y}, \tilde{s}))$. Then rewrite

$$z = (\tilde{y} - \tilde{x}; \tilde{s} - s).$$

We note that $\tilde{s} - s \leq 0$; moreover, by T.6. if $\tilde{y} - \tilde{x} > 0$, then $\tilde{s} - s < 0$. Hence,

Z is a closed convex cone that does not contain any $u > 0$.

Using Theorem 3.6 of Nikaido (1968), there is some $p \gg 0$ such that $p \cdot z \leq 0$ for all $z \in Z$. This completes the proof of lemma 1.■

Since $p \gg 0$, we can normalize and take $p = (q, 1)$. Then $p \cdot z = [q \cdot (\tilde{y} - \tilde{x}) + \tilde{s} - s] \leq 0$. Let $(\mathbf{x}, \mathbf{y}, \mathbf{s}; \mathbf{c})$ =$(x_t, y_{t+1}, s_t; c_{t+1})$ be any program (given $X \gg 0$ and $S > 0$). Then,

$$c_1 = \tilde{y}_1 - \tilde{x}_1, \text{ and, } ((X, S), (\tilde{y}_1, s_1)) \in \mathcal{T} \ .$$

Hence,

$$q \cdot (\tilde{y}_1 - X) + s_1 - S \leq 0,$$

which leads to

$$q \cdot \tilde{y}_1 \leq q \cdot X + S - s_1.$$

Now,

$$q \cdot c_1 = q \cdot (\tilde{y}_1 - \tilde{x}_1) = q \cdot \tilde{y}_1 - q \cdot \tilde{x}_1 \leq q \cdot X + S - s_1 - q \cdot \tilde{x}_1 \tag{15}$$

Similarly,

$$c_2 = \tilde{y}_2 - \tilde{x}_2, \text{ and } ((\tilde{x}_1, s_1), (\tilde{y}_2, s_2)) \in \mathcal{T}$$

Hence,

$$q \cdot (\tilde{y}_2 - \tilde{x}_1) + s_2 - s_1 \leq 0$$

which leads to

$$q \cdot \tilde{y}_2 \leq q \cdot \tilde{x}_1 + s_1 - s_2.$$

Now,

$$q \cdot c_2 = q \cdot (\tilde{y}_2 - \tilde{x}_2) = q \cdot \tilde{y}_2 - q \cdot \tilde{x}_2 \leq q \cdot \tilde{x}_1 + s_1 - s_2 - q \cdot \tilde{x}_2 \tag{16}$$

From (15) and (16)

$$q \cdot c_1 + q \cdot c_2 \leq q \cdot X + S - s_2 - q \cdot \tilde{x}_2.$$

Continuing this process we find:

$$\sum_{t=1}^{T} (q \cdot c_t) \leq q \cdot X + S - s_T - q \cdot \tilde{x}_T \leq q \cdot X + S.$$

This implies that

$$\sum_{t=1}^{\infty}(q\cdot c_t)=\lim_{T\to\infty}[\sum_{t=1}^{T}(q\cdot c_t)]\leq q\cdot X+S.$$

In particular,

$$\lim_{t\to\infty}(q\cdot c_t)=0.$$

Let $\alpha=\min_k(q_k)$. Then, from (14), $(q\cdot c_t)\geq\alpha\|c_t\|\geq 0$. Hence,

$$\lim_{t\to\infty}\|c_t\|=0.$$

■

The problem of identifying "substitution" conditions on the technology that guarantee the possibility of maintaining a strictly positive level of consumption was considered by Solow (1974) in a model with one producible good (that can be either consumed or used as an input in the production process) and one resource (an important input) and a Cobb-Douglas production function. We follow Mitra's subsequent investigation informally (the interested reader is referred to Mitra , 1978b and Cass and Mitra, 1991 for a rigorous presentation). Let $G:R_{+}^{2}\to R_{+}$ be the net output function: $G(x,r)$ is the net output of the producible good when $x(\geq 0)$ is the input of the producible good ("capital") and r is the quantity of the resource. Assuming no depreciation, write $F(x,r)=G(x,r)+x$ as the gross output function. With a slight abuse of notation (we are writing x and y to indicate the input and output of the single producible good and dispensing with the symbols $\tilde{x}$, $\tilde{y}$ used earlier) we can describe the technology as:

$$\mathcal{T}=\{((x,s),(y,\hat{s}))\in R_{+}^{2}\times R_{+}^{2}:\ 0\leq y\leq F(x,r),\ \text{where } 0\leq r\leq s-\hat{s}.$$

Given the initial stock of the producible good $X>0$, of the resource $S>0$. A program is a sequence $(\mathbf{x}, \mathbf{y}, \mathbf{s}; \mathbf{c}) = (x_t, y_{t+1}, s_t; c_{t+1})$ satisfying

$$\begin{aligned} x_0 &= X,\ s_0 = S, \\ ((x_t, s_t), (y_{t+1}, s_{t+1})) &\in \mathcal{T} \text{ for } t \geq 0; \\ c_{t+1} &= y_{t+1} - x_{t+1} \geq 0 \text{ for } t \geq 0. \end{aligned} \tag{17}$$

The following assumption on G are made:

G.1. *G is continuous on R_+^2; it is homogeneous of degree one, concave, twice continuously differentiable for $(x, r) \gg 0$, $G_x > 0$, $G_r > 0$.*

A program from $(X, S) \gg 0$ is said to *sustain a positive consumption level* if $\inf_{t \geq 1} c_t > 0$.

For any positive scalar $d > 0$, let $Q(d) = \{x \geq 0 : G(x, r) = d \text{ for some } r \geq 0\}$.The isoquant function corresponding to the output level d is $i(d) : Q(d) \to R_+$, i.e., for $x \in Q(d)$, $G(x, i(x)) = d$. The usual smoothness conditions $i' < 0$, $i'' < 0$ hold. For any $\theta > 0$ such that $\theta \in Q(d)$, $\int_\theta^{\theta+L} i(x)dx$ is non-decreasing in L. Hence, as $L \to \infty$, $\int_\theta^{\theta+L} i(x)dx$ either converges to some finite limit or diverges to ∞.The integral

$$\int_\theta^\infty i(x)dx$$

is called θ-area under the d-isoquant. If, for every $d > 0$, and every $\theta \in Q(d)$, the θ-area under the d-isoquant is finite, we define the production function G to be "regular". One of the main results due to Mitra is: *if G* is "regular", there is a *program from $(X, S) \gg 0$ that sustains a positive consumption level.* Indeed, under some additional conditions (satisfied, for example, when G has the Cobb-Douglas functional form), it is also true that *if G is not "regular"*, there is *no* program from $(X, S) \gg 0$ that sustains a positive consumption level. Take the case $G(x, r) = x^\alpha r^\beta$ where $\alpha + \beta = 1$ (α, β positive). Then,for any $d > 0$, let $\theta \in Q(d)$ (here choose any $\theta > 0$); then the $d-$isoquant is given by $i(x) = (d^{1/\beta})/x^\nu$where $\nu = \alpha/\beta$. So, the θ-area under the d-isoquant is finite if and only if $\nu > 1$, i.e., $\alpha > \beta$. Thus, a positive consumption level is sustainable if and only if $\alpha > \beta$. This was obtained by Solow (1974).

4 A Model of Optimistic Transition

A transition from a technology that uses an exhaustible resource as an essential input to one in which the exhaustible resource is not used has been studied by a number of writers (see Smith, 1974 or Heal, 1972 and the references cited there). In this section, we consider an economy with an exhaustible resource which can be used as an input in a production function $F(.,.)$. Let the total supply of this resource be $S > 0$. At the "beginning" of each period t the economy also gets a quantity $l_t \equiv \bar{l} > 0$ of another factor of production. Let $q_{1t} = F(r_t, l_{1t})$ be the output (appearing at the "end" of period t) when the input pair is $(r_t, l_{1t}) \geq 0$. Assume that:

F.1. $F : R_+^2 \rightarrow R_+$ *is continuous.*

F.2. $F(0,l) = F(r,0) = 0$, *in other words, both inputs are essential in production.*

F.3. F is nondecreasing in each argument, and $(r,l) \gg (r',l')$ implies that $F(r,l) > F(r',l')$.

There is an alternative technology producing the same output, given by $q_{2t} = \theta \cdot l_{2t}$ where:

F.4. $\theta > 1$.

Observe that this technology does not need the exhaustible resource as an input. We shall call this a "green technology".

Given $\langle S; l_t \equiv \bar{l} \rangle$ a *program* is a *non-negative* sequence $\langle q_{1t},\ q_{2t}; r_t, l_{1t}; l_{2t} \rangle$ satisfying:

$$\begin{aligned} q_{1t} &= F(r_t, l_{1t});\ q_{2t} = \theta \cdot l_{2t} \text{ for all } t \geq 0 \\ l_{1t} + l_{2t} &= \bar{l} \text{ for all } t \geq 0 \\ \sum_{t=0}^{\infty} r_t &\leq S. \end{aligned}$$

Thus, a program is a specification of decisions on resource allocation that can be informally described as follows: at the "beginning" of period $t = 0$, it starts with the stock $S > 0$, and the supply of $l_0 = \bar{l} > 0$. It decides on the (non-negative) quantities $\langle l_{10}, l_{20}; r_0 \rangle$ subject to the constraint $l_{10} + l_{20} = \bar{l}$ and $r_0 \leq S$. As a result, the outputs $(q_{10},\ q_{20})$ appear at the end of that period. At the beginning of the next period $t = 1$, it starts with the stock $S - r_0 \geq 0$, and the supply of $l_1 = \bar{l} > 0$, and the decision making task is repeated. Note that if an any period, $l_{2t} = 0$, the corresponding $q_{2t} = 0$. similarly, if $l_{1t} = 0$, or $r_t = 0$ (or, both), the corresponding $q_{1t} = 0$.

We now introduce a notion of optimality. As before, we denote the discount factor by δ. Given $(S, \bar{l}) \gg 0$, and $0 < \delta < 1$, a program $\mathbf{x} = \langle q_{1t}^*, q_{2t}^*; r_t^*, l_{1t}^*; l_{2t}^* \rangle$ is *optimal* if

$$\sum_{t=0}^{\infty} \delta^t (q_{1t}^* + q_{2t}^*) \geq \sum_{t=0}^{\infty} \delta^t (q_{1t} + q_{2t})$$

for all programs $\langle q_{1t},\ q_{2t}; R_t, L_{1t}; L_{2t} \rangle$. Informally, a program is optimal if it generates the maximum discounted sum of outputs among all programs.

4.1 Analysis

The first thing we establish is that there exist an optimal program. Let us note some boundedness properties. As always, we are given $(S, l_t = \bar{l})$. Write $B = F(S, \bar{l})$ and $\gamma = \theta \cdot \bar{l}$. Then for any program $\mathbf{x}$,

$$v(\mathbf{x}) = \sum_{t=0}^{\infty} \delta^t \left(q_{1t} + q_{2t}\right) \leq \sum_{t=0}^{\infty} \delta^t \left(B + \gamma\right) \equiv \frac{B + \gamma}{1 - \delta}.$$

Thus, $\alpha \equiv \sup \left\{v(\mathbf{x}) : \mathbf{x} \text{ is a program given } (S, l_t = \bar{l})\right\}$ is well-defined.

Proposition 12 *There exists an optimal program* $\mathbf{x}^* = \langle q_{1t}^*, q_{2t}^*; r_t^*, l_{1t}^*; l_{2t}^* \rangle$.

Proof. We can adapt the proof of Proposition 2 above. ∎

Our next proposition is the principal result on the eventual switch to the green technology.

Proposition 13 *Let* $\mathbf{x}^* = \langle q_{1t}^*,\ q_{2t}^*; r_t^*, l_{1t}^*; l_{2t}^* \rangle$ *be an optimal program. Then there is some finite* T *such that* $l_{2t}^* > 0$, *for all* $t \geq T$.

Proof. If the claim is *not* true, then there is an infinite subsequence of periods t_n such that $l^*_{2t_n} = 0$ and $F(r^*_{t_n}, \bar{l}) = q^*_{1t_n} > 0$ for all t_n. Since an optimal program is a (feasible) program, $\sum_t r^*_t \leq S$. This means that given any $\epsilon^/ > 0$, there is some $T(\epsilon^/)$ such that $0 < r^*_t < \epsilon^/$ for all $t \geq T(\epsilon^/)$. In particular, for all $t_n \geq T(\epsilon^/)$, it is true that $0 < r^*_{t_n} < \epsilon^/$.

By continuity of F and the assumption that $F(0, \bar{l}) = 0$, there is some $\epsilon > 0$ sufficiently small such that $F(r, \bar{l})$ is less than some $\nabla > 0$, where $\nabla < (\theta \cdot \bar{l})$, for all $0 \leq r < \epsilon$. Consider now a period $t_n \geq T(\epsilon)$. Then, $F(r^*_{t_n}, \bar{l}) < \nabla$. Hence, $q^*_{1t_n} < \nabla$. Consider an alternative program that agrees with our assumed optimal program for all periods except period t_n, but uses only the green technology in period $t_n \geq T$. The total output in period t_n is $(\theta \cdot \bar{l})) > \nabla > q^*_{1t}$. Clearly, the total output from this alternative program exceeds that of the optimal program, and this contradicts the optimality of $\mathbf{x}^*$, and establishes our claim. ∎

4.2 Examples

We consider two examples in which technology 1 takes parametric functional forms: one with a perfect complements technology and one with a Cobb-Douglas technology.

Example 3: Assume that the production function which uses the resource is a perfect complements technology $F(r_t, l_{1t}) = \min\{r_t, al_{1t}\}$, and $S > 0$. If technology 1 is used in some period t then it cannot be that $r^*_t < al^*_{1t}$ or else some labor used in technology 1 is not productive and it can be productive in technology 2. If in the optimal program $r^*_t > al^*_{1t}$ then there is also an optimal program with $r^{**}_t = al^*_{1t}$ leaving the remaining $r^*_t - al^*_{1t}$ unused. We therefore focus on the case where in every period, $r^*_t = al^*_{1t}$.

We consider two cases depending on the relative productivity of labor in the two available technologies: (i) $\theta > a$ and (ii) $\theta < a$. We show that the resource is not used at all in case (i) where the green technology is very productive. But, in case (ii) the entire stock of the resource is exhausted.

We will show that for a sufficiently large initial stock of the exhaustible resource, there is a number of initial periods in which only the first technology is used. Then, there is a switch to the green technology, with at most one period of overlapping use of the two technologies. We state these findings more formally.

(i) If $F(r_t, l_{1t}) = \min\{r_t, al_{1t}\}$ *and* $\theta > a$ *then only technology 2 is used.*

(ii) If $F(r_t, l_{1t}) = \min\{r_t, al_{1t}\}$ *and* $\theta < a$ *then in any optimal program all of the resource is used,* $\sum_t r_t^* = S$*. Moreover, only technology 1 will be used for* T *periods where* $T = \left\lfloor \frac{S}{a\bar{l}} \right\rfloor$ *(i.e., the integer part of* $\frac{S}{a\bar{l}}$*). In period* $T+1$ *the remaining quantity of the resource (if any) is used* $r_{T+1} = S - Ta\bar{l}$*,* $l_{1T+1} = \frac{S - Ta\bar{l}}{a}$ *and for all* $t > T+1$*, only technology 2 is used* $r_{T+1} = l_{1T+1} = 0$*.*

Proof. (i) We assume here that $\theta > a$. Suppose in an optimal program $\langle r_t^*, l_{1t}^*; l_{2t}^* \rangle$ technology 1 is used in some period t. Recall that $r_t^* = al_{1t}^*$. Consider an identical program except that we change allocations in period t by moving $\varepsilon \in (0, l_{1t}^*)$ units of l from technology 1 to technology 2. Production does not change in any other period. In period t the change in production is

$$\begin{aligned} & \left[\min\{r_t^*, a(l_{1t}^* - \varepsilon)\} + \theta\left(\bar{l} - l_{1t}^* + \varepsilon\right)\right] - \left[\min\{r_t^*, al_{1t}^*\} + \theta\left(\bar{l} - l_{1t}^*\right)\right] \\ = & \left[a(l_{1t}^* - \varepsilon) + \theta\left(\bar{l} - l_{1t}^* + \varepsilon\right)\right] - \left[al_{1t}^* + \theta\left(\bar{l} - l_{1t}^*\right)\right] \\ = & (\theta - a)\varepsilon > 0. \end{aligned}$$

This is a contradiction to the optimality of the initial program.

(ii) Assume now that $\theta < a$. Suppose by way of contradiction that $\sum_t r_t^* < S$. Since $r_t^* \to 0$, there is at least one period t in which $r_t^* < a\bar{l}$. In period t technology 2 must be used, $l_{2t}^* > 0$, or else some labor is not productive in technology 1 but it could be in technology 2. Recall that $r_t^* = al_{1t}^*$. Consider the effect of adding $\varepsilon \in \left(0, \min\{S - \sum_t r_t^*, al_{2t}^*\}\right)$ of the remaining resource to period t production and moving $\frac{\varepsilon}{a}$ of the labor in period t from technology 2 to technology 1, making no other changes to the program. The change in production in this period is

$$
\begin{aligned}
& \left[\min\left\{r_t^* + \varepsilon, a\left(l_{1t} + \frac{\varepsilon}{a}\right)\right\} + \theta\left(l_{2t}^* - \frac{\varepsilon}{a}\right)\right] - \left[\min\left\{r_1^*, al_{1t}^*\right\} + \theta l_{2t}^*\right] \\
= & \left[a\left(l_{1t}^* + \frac{\varepsilon}{a}\right) + \theta l_{2t}^* - \theta\frac{\varepsilon}{a}\right] - \left[al_{1t}^* + \theta l_{2t}^*\right] \\
= & \left(a - \theta\right)\frac{\varepsilon}{a} > 0.
\end{aligned}
$$

This is a contradiction to the optimality of the initial program. This establishes that $\sum_t r_t^* = S$.

We next show that if $S > a\bar{l}$, then $l_{10}^* = \bar{l}$, i.e., in period 0 only technology 1 is used. Suppose by way of contradiction that in an optimal program $\langle r_t^*, l_{1t}^*; l_{2t}^*\rangle$, in period 0, $l_{10}^* < \bar{l}$ and $l_{20}^* > 0$. In this period, $r_0^* = al_{10}^* \geq 0$. We know that all of S is used in the optimal program. Because $S > a\bar{l}$, and for all t, $r_t^* = al_{1t}^* \leq a\bar{l} < S$ it must be that S is used for at least two periods. Hence, there exists at least one more period where technology 1 is used. Let $t > 0$ be a period where the resource is used so that $r_t^* = al_{1t}^* > 0$. Now consider moving $\varepsilon \in (0, \min\{r_t^*, al_{20}^*\})$ of the resource from period t to period 0 adjusting labor in both periods accordingly. The new program, denoted by ** is everywhere the same as the proposed optimal program only that $r_0^{**} = r_0^* + \varepsilon$ and $r_t^{**} = r_t^* - \varepsilon$. Also $l_{10}^{**} = l_{10}^* + \frac{\varepsilon}{a}$ and $l_{20}^{**} = l_{20}^* - \frac{\varepsilon}{a}$ and $l_{1t}^{**} = l_{1t}^* - \frac{\varepsilon}{a}$ and $l_{2t}^{**} = l_{2t}^* + \frac{\varepsilon}{a}$. Clearly the new program is still feasible. Production does not change in any other periods except in periods 0 and t. In these periods the change in production is

$$
\begin{aligned}
& \left[\min\left\{r_0^* + \varepsilon, a\left(l_{10}^* + \frac{\varepsilon}{a}\right)\right\} + \theta\left(l_{20}^* - \frac{\varepsilon}{a}\right)\right] \\
& + \delta^t\left[\min\left\{r_t^* - \varepsilon, a\left(l_{1t}^* - \frac{\varepsilon}{a}\right)\right\} + \theta\left(l_{2t}^* + \frac{\varepsilon}{a}\right)\right] \\
& - \left[\min\left\{r_0^*, al_{10}^*\right\} + \theta l_{20}^*\right] - \delta^t\left[\min\left\{r_t^*, al_{1t}^*\right\} + \theta l_{2t}^*\right] \\
= & \; al_{10}^* + \varepsilon + \theta l_{20}^* - \frac{\theta\varepsilon}{a} - al_{10}^* - \theta l_{20}^* \\
& + \delta^t\left[al_{1t}^* - \varepsilon + \theta l_{2t}^* + \frac{\theta\varepsilon}{a} - al_{1t}^* - \theta l_{2t}^*\right] \\
= & \left[1 - \delta^t\right]\frac{a - \theta}{a}\varepsilon > 0.
\end{aligned}
$$

This contradicts the optimality of the initial program, and implies that only technology 1 is used as long as the remaining resource is high enough, and then there is a switch to the green technology. ■

Example 4: Consider technology 1 which is a Cobb-Douglas production function $F(r,l) = r^{\alpha} l^{\beta}$ where $\alpha, \beta > 0$. We will distinguish in this example between production functions with $\beta < 1$, or $\beta \geq 1$, which determine whether the marginal productivity of labor is decreasing, constant or increasing with labor. With diminishing marginal productivity of labor in technology 1, we find the the entire quantity of the resource would be exhausted. With increasing marginal product, if the green technology is productive enough, it is possible that only the green technology is used. We prove the following claims:

Assume $F(r,l) = r^{\alpha} l^{\beta}$. *(i) If* $\beta < 1$ *then in an optimal program, all of the resource must be used. (ii) If* $\beta \geq 1$ *and* $S^{\alpha} \bar{l}^{\beta-1} > \theta$ *then in an optimal program, all of the resource must be used. (iii) If* $\beta \geq 1$ *and* $S^{\alpha} \bar{l}^{\beta-1} < \theta$ *then in an optimal program, only technology 2 is used.*

Proof. (i) Assume $\beta < 1$. Suppose first that only technology 2 is used. Consider moving $\varepsilon > 0$ of labor in period 0 from technology 2 to technology 1, and using it with the entire stock of the resource S. Keep all other periods unchanged. Then in period 0 the change in production is

$$S^{\alpha} \varepsilon^{\beta} + \theta \cdot \left(\bar{l} - \varepsilon\right) - \theta \cdot \bar{l} = \left(S^{\alpha} \varepsilon^{\beta} - \theta \varepsilon\right) = \left(S^{\alpha} - \theta \varepsilon^{1-\beta}\right) \varepsilon^{\beta}.$$

Taking $\varepsilon \in \left(0, \min\left\{\left(\frac{S^{\alpha}}{\theta}\right)^{\frac{1}{1-\beta}}, \bar{l}\right\}\right)$ we get $S^{\alpha} - \theta \varepsilon^{1-\beta} > 0$ and thus an increase in production. This contradicts optimality of a program that uses only technology 2. Hence, at least some of the resource must be used in any optimal program. Now because F is strictly increasing in r for all $l > 0$, it must be that all of the resource is used. Else, increase output by adding the remaining r in a period where $l_{1t} > 0$.

(ii) If $\beta \geq 1$ and $S^{\alpha} \bar{l}^{\beta-1} > \theta$. Suppose first that only technology 2 is used. Consider moving $\bar{l}$ in period 0 from technology 2 to technology 1, and using it with the entire stock of the resource S. Keep all other periods unchanged. Then in period 0 the change in production is

$$S^{\alpha}\bar{l}^{\beta} - \theta \cdot \bar{l} = \left(S^{\alpha}\bar{l}^{\beta-1} - \theta\right) \cdot \bar{l} > 0.$$

This contradicts optimality of a program that uses only technology 2. Hence, at least some of the resource must be used in any optimal program. Again, since F is strictly increasing in r for all $l > 0$, it must be that all of the resource is used. Else, increase output by adding the remaining r in a period where $l_{1t} > 0$.

(iii) The third case we need to consider is $\beta \geq 1$ and $S^{\alpha}\bar{l}^{\beta-1} < \theta$. Assume by way of contradiction there is a period t in which technology 1 is used. Consider moving the labor used in this period to technology 2. The change in production is:

$$\theta \cdot \bar{l} - \left[r_t^{\alpha} l_{1t}^{\beta} + \theta \cdot \left(\bar{l} - l_{1t}\right)\right] = \left(\theta \cdot l_{1t} - r_t^{\alpha} l_{1t}^{\beta}\right) > \left(S^{\alpha}\bar{l}^{\beta-1} - r_t^{\alpha} l_{1t}^{\beta-1}\right) l_{1t} \geq 0.$$

This contradicts the optimality of the optimal program. ■

5 Conclusion

It is too much to expect that simple *deterministic* models can capture the complexity of the process of switching from a "conventional" technology essentially dependent on exhaustible resources as inputs to one in which such resources impose no serious constraint on production. At many steps in an adoption process, decision making is characterized by incomplete information about an emerging alternative technology (about productivity, safety, environmental impact, etc.). The substantial literature on adoption of a new technology by an *individual* firm deals with uncertainty from several perspectives (see Hoppe, 2002). But this literature does not recognize the tension created by the possibilities of exhaustion or extinction. However, any formal analysis of stochastic models to deal with uncertainty involves concepts and techniques other than those we have relied upon in this exposition.

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